a, d similar, but the x component for d larger

MATH 5C - TEST 4

Chapter 16

Spring 24 v1

100 POINTS NAME: ____________________

Show all work neatly with complete explanations.

(1) Match the equation to the vector field plot. Vector fields have been uniformly scaled in order to be seen more clearly. Two plots do not have a match. (8pts)

(8pts)

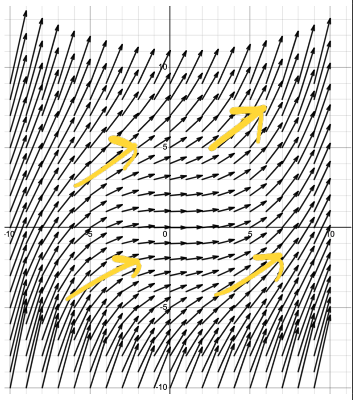
a) $\vec{F}(x,y)=\langle x,x\rangle$ ____6____

b) $\vec{F}(x,y)=\langle 4\cos(x+y),x\rangle$ ____2____

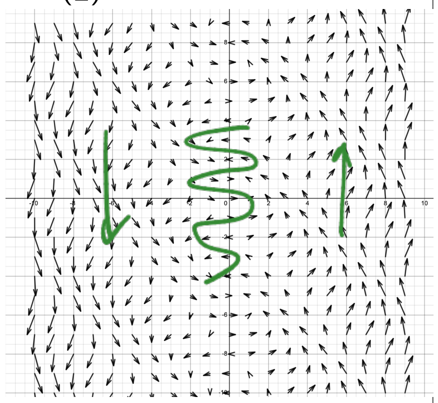
c) $\vec{F}(x,y)=\left\langle 10,\frac{1}{4}(x^2+y^2)\right\rangle$ ____1____

d) $\vec{F}(x,y)=\langle 6x,x\rangle$ ____3____

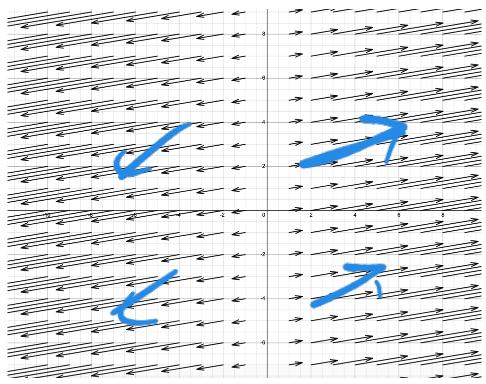
(1)



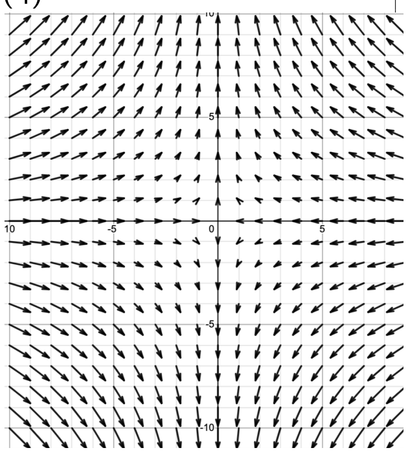
(2)



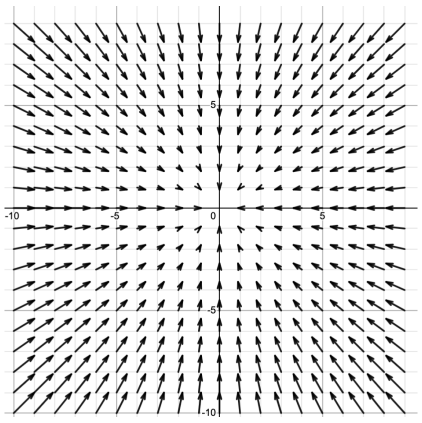
(3)



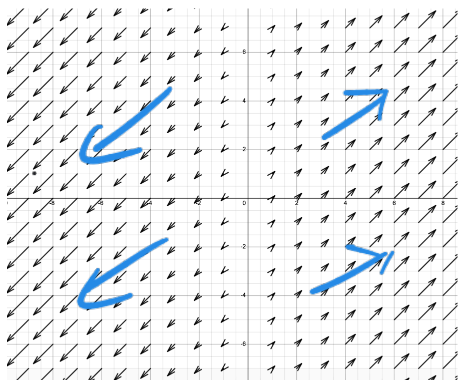
(4)



(5)



(6)



(2) Find the work done by the vector field $\vec{F}(x,y,z) = <2x, -x-z,\ y-x>$ in moving an object along the curve $\vec{r}(t) = \langle t^2,\ t^2-1,\ 3\rangle;\quad 0 \le t \le 3$ using any appropriate method. Directly

(10 points)

$\vec{F}(t) = \langle 2t^2, -t^2-3,\ t^2-1-t^2 \rangle$

$\vec{r}\,' = \langle 2t,\ 2t,\ 0 \rangle$

$\vec{F} \cdot \vec{r}\,' = 4t^3 - 2t^3 - 6t = 2t^3 - 6t$

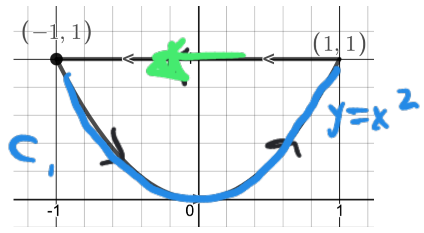
$$\text{work} = \int_0^3 (2t^3 - 6t)\,dt$$

$$= \frac{1}{2}t^4 - 3t^2 \Big]_0^3$$

$$= \frac{81}{2} - 27 = \boxed{\frac{27}{2}}$$

(3) Verify Green's Theorem is true for the line integral $\oint_C x^2\,dx - x^3 y\,dy$ where C is the piecewise curve consisting of the parabola $y=x^2$ for (-1,1) to (1,1) and the line segment from (1,1) to (-1,1). (That is, do the problem directly and then using Green's theorem). (24 points)

$F = \langle x^2, -x^3 y \rangle$



directly

C_1 $\begin{cases} x=t \\ y=t^2 \end{cases}$ $-1 \le t \le 1$

$\vec{F} = \langle t^2, -t^5 \rangle$

$\vec{r}\,' = \langle 1, 2t \rangle$

$\int_{-1}^{1} (t^2 - 2t^6)\,dt$

$\frac{t^3}{3} - \frac{2}{7}t^7 \Big]_{-1}^{1}$

$2\left(\frac{1}{3} - \frac{2}{7}\right) = \frac{2}{21}$

C_2 $x = 1-2t$
$y = 1$
$0 \le t \le 1$

$\vec{F} = \langle (1-2t)^2, (1-2t)^3 \rangle$

$\vec{r}\,' = \langle -2, 0 \rangle$

$\int_0^1 -2(1-2t)^2\,dt$ $u=1-2t$ $du=-2dt$

$\frac{(1-2t)^3}{3} \Big]_0^1$

$-\frac{1}{3} - \frac{1}{3} = -\frac{2}{3}$

$\Rightarrow \frac{2}{21} + \frac{-2}{3} = \frac{-12}{21} = -\frac{4}{7}$

Greens

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = -3x^2 y - 0$

$\int_C \vec{F} \cdot d\vec{r} = \iint_D -3x^2 y\,dA = \int_{-1}^{1}\int_{x^2}^{1} -3x^2 y\,dy\,dx = \int_{-1}^{1} -3x^2 \frac{y^2}{2}\Big]_{x^2}^{1} dx$

$= \int_{-1}^{1} -\frac{3}{2}x^2(1-x^4)\,dx = -\frac{3}{2}\int_{-1}^{1}(x^2 - x^6)\,dx = -\frac{3}{2}\left(\frac{x^3}{3} - \frac{x^7}{7}\right]_{-1}^{1}$

$-\frac{3}{2} \cdot \frac{8}{21} = \boxed{-\frac{4}{7}}$

(4) Evaluate the flux integral $\iint_S \vec{F} \bullet d\vec{S} = \iint_S \vec{F} \bullet \vec{n}\, dS$ where $\vec{F}(x,y,z) = \langle -y, x, 3z \rangle$ and S the portion of the cone $z = \sqrt{x^2+y^2}$ below z=4, positively oriented. (Note: this is NOT a closed surface)

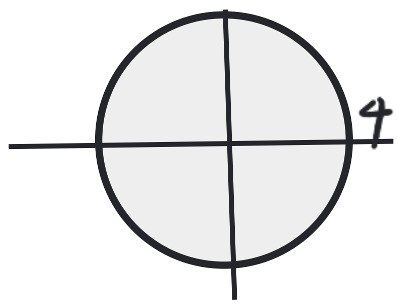
(12 points)

S: $z = g(x,y) = \sqrt{x^2+y^2}$

$$z - g(x,y) = \underbrace{z - \sqrt{x^2+y^2}}_{G(x,y,z)} = 0$$

$$\vec{\nabla} G = \left\langle \frac{-x}{\sqrt{x^2+y^2}}, \frac{-x}{\sqrt{x^2+y^2}}, 1 \right\rangle$$

$$\vec{F} = \langle -y, x, 3\sqrt{x^2+y^2} \rangle$$



$$\vec{F} \cdot \vec{\nabla} G = 3\sqrt{x^2+y^2}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{\nabla} G\, dA$$

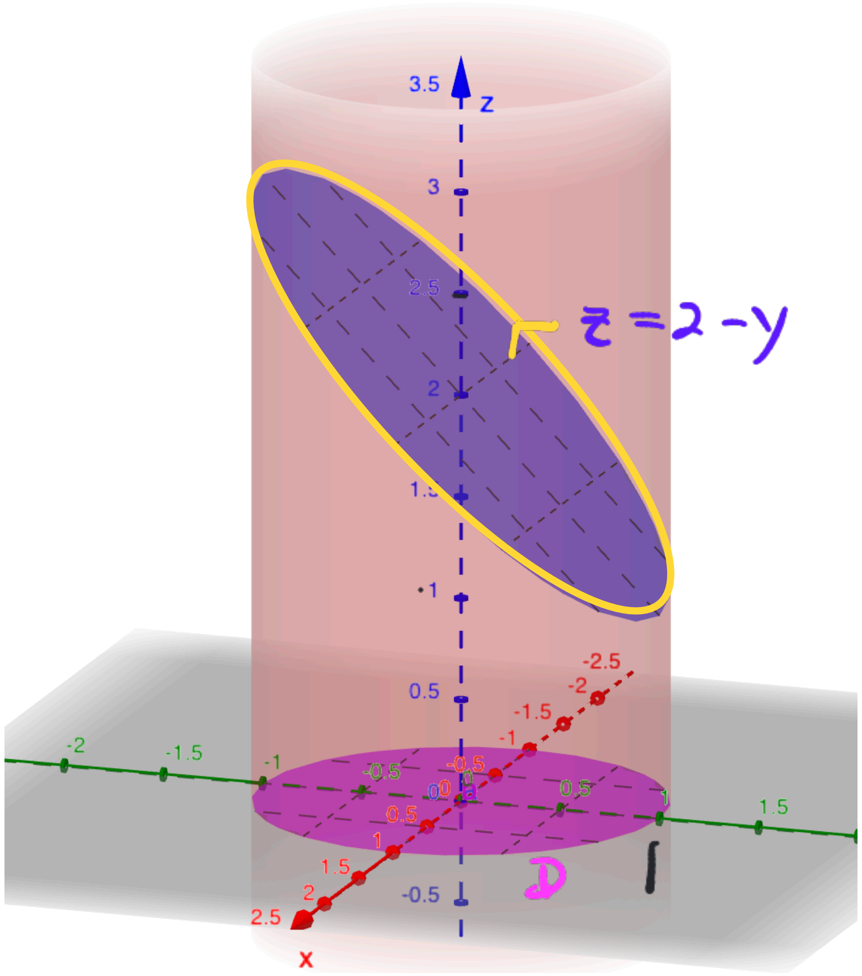
$$= \iint_D 3\sqrt{x^2+y^2}\, dA$$

$$= \int_0^{2\pi} \int_0^4 3r^2\, dr\, d\theta$$

$$= 2\pi \cdot 4^3 = \boxed{128\pi}$$

(5) Given $\vec{F}(x,y,z) = -y^2\,\vec{i} + x\,\vec{j} + z^2\,\vec{k}$ and C is the curve of intersection of the plane $z = 2 - y$ and the cylinder $x^2 + y^2 = 1$ oriented in the positive direction, find $\int_C \vec{F}\bullet d\vec{r}$ using any appropriate method. Explain

(12 points)



Using Stokes'

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$$

$$= \iint_D (\text{curl}\,\vec{F}\cdot\vec{\nabla} G)\,dA$$

Compute curl $\vec{F}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \boxed{\langle 0, 0, 1+2y\rangle}$$

$$\text{curl}\,\vec{F}\cdot\vec{\nabla}G = 1+2y$$

surface

$S:\ z = 2-y$

$\underbrace{z+y-2}_{G} = 0$

$\vec{\nabla}G = \langle 0, 1, 1\rangle$

So

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D (1+2y)\,dA$$

$$= \int_0^{2\pi}\int_0^1 (r + 2r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{2} + \frac{2}{3}\sin\theta\right)d\theta$$

$$= \boxed{\pi}$$

Directly

parameterize C

$\vec{r} = \langle \cos t, \sin t, 2-\sin t\rangle$

$\vec{r}\,' = \langle -\sin t, \cos t, -\cos t\rangle$

$\vec{F} = \langle -\sin^2 t, \cos t, (2-\sin t)^2\rangle$

$\vec{F}\cdot\vec{r}\,' =$

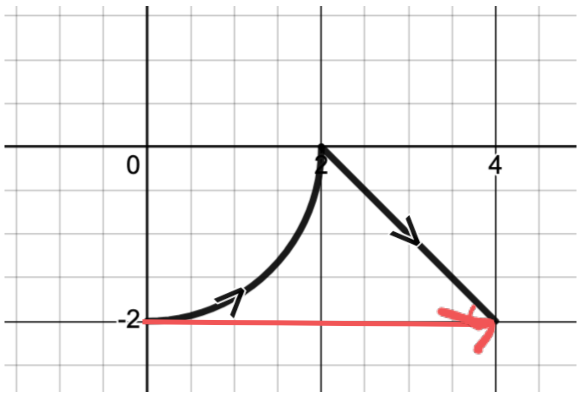
$\sin^3 t + \cos^2 t - \cos t(2-\sin t)^2$

$\vdots$

(22 points)

(6) Given the vector field $\vec{F}(x,y)=\langle 2x+y^2, 2xy+1\rangle$ piecewise smooth path C given by a quarter circle of radius 2, traveled from (0,-2) to (2,0), followed a line segment from (2,0) to (4,-2) as shown ,

$16+16-2$



$\frac{\partial f}{\partial x} = 2x+y^2$

$f(x,y) = x^2 + xy^2 + c(y)$

$\frac{\partial f}{\partial y} = 2xy + c'(y) = 2xy+1$

$c'(y) = 1$

$c(y) = y + c$

a) Find the potential function $f(x,y)$ such that $\vec{\nabla} f(x,y) = \vec{F}(x,y)$ and use it to compute $\int_C \vec{F} \bullet d\vec{r}$

$\boxed{f(x,y) = x^2 + xy^2 + y + C}$

$\int_C \vec{F}\cdot d\vec{r} = f(4,-2) - f(0,-2)$ by Fundamental Thm of Line Int.

$= 30 + 2 = \boxed{32}$

b) Find $\int_C \vec{F} \bullet d\vec{r}$ using a different method. Explain

Can do it directly in two pieces, or choose a simpler path since $\vec{F}$ is conservative.

segment from $(0,-2) \to (4,-2)$

$x = 4t$
$y = -2$
$0 \le t \le 1$

$\vec{F} = \langle 8t+4, -16t+1\rangle$

$\vec{r}\,' = \langle 4, 0\rangle$

$\int_0^1 4(8t+4)\,dt = 16t^2 + 16t\Big]_0^1 = \boxed{32}$

(7) Evaluate the flux integral $\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS$ where $\vec{F}(x,y,z) = \langle x, y, z^2 - 1 \rangle$ and S closed, positively oriented surface of the solid bound by the cylinder $x^2 + y^2 = 25$ and the planes $z = 0; z = 1$

(12 points)

Divergence Theorem

$$\operatorname{div} \vec{F} = 1 + 1 + 2z = 2 + 2z$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

$$= \int_0^{2\pi} \int_0^5 \int_0^1 (2 + 2z)\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^5 2z + z^2 \Big]_0^1 \, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^5 3r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{3}{2} r^2 \Big]_0^5$$

$$= 2\pi \, \frac{75}{2} = \boxed{75\pi}$$

MATH 5C - TEST 4

Chapter 16 v2

Spring 2024

100 POINTS NAME: ____________________

Show all work neatly with complete explanations.

(1) Match the equation to the vector field plot. Vector fields have been uniformly scaled in order to be seen more clearly. Two plots do not have a match. (8pts)

a) $\vec{F}(x,y)=\langle y^2,-x+y\rangle$ __3__

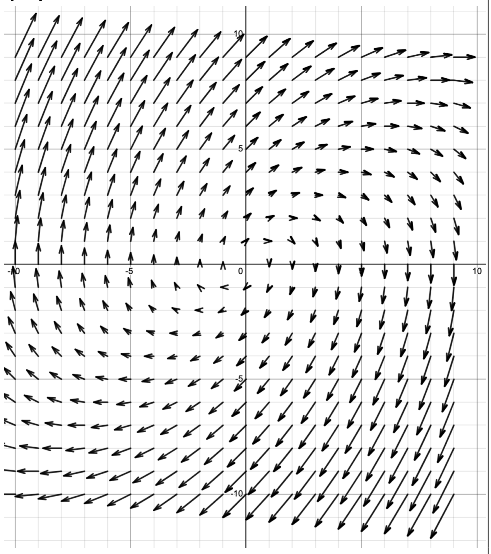
b) $\vec{F}(x,y)=\langle y^3,x\rangle$ __5__

c) $\vec{F}(x,y)=\langle 6x,x\rangle$ __6__

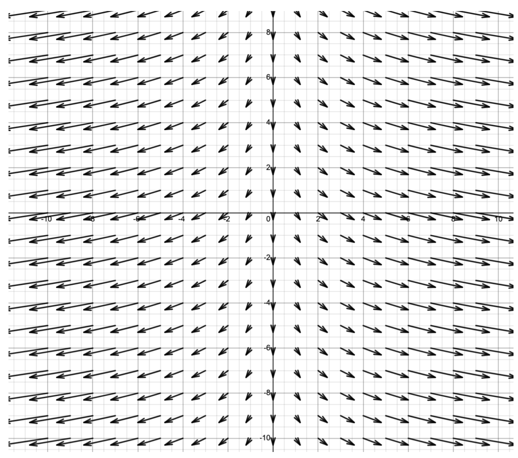
d) $\vec{F}(x,y)=\langle y,x\rangle$ __4__

b, d similar but x coordinate larger for y>1

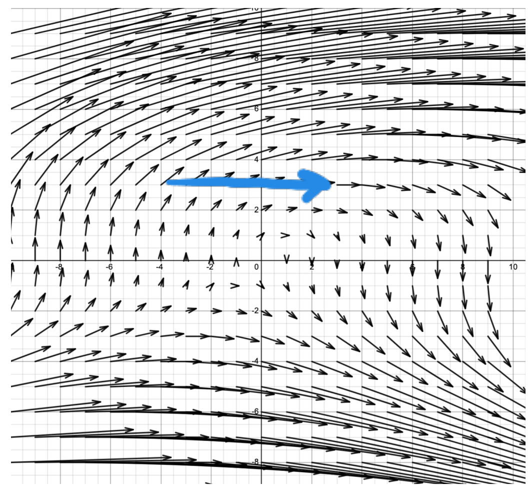
(1)



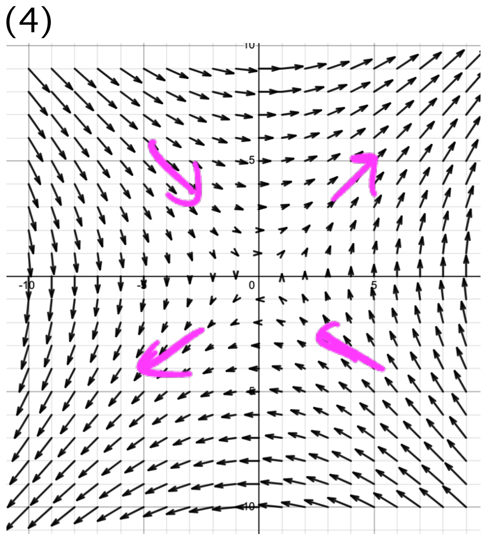
(2)



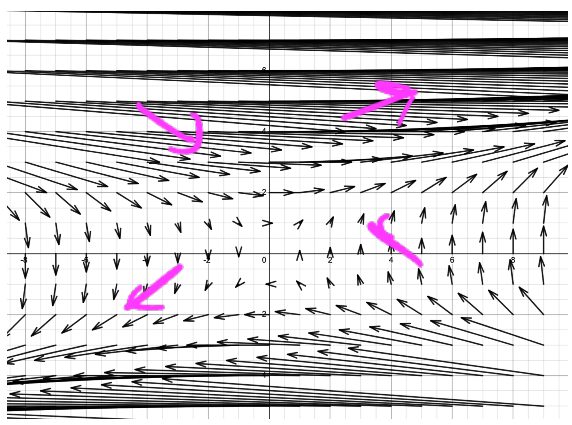
(3)



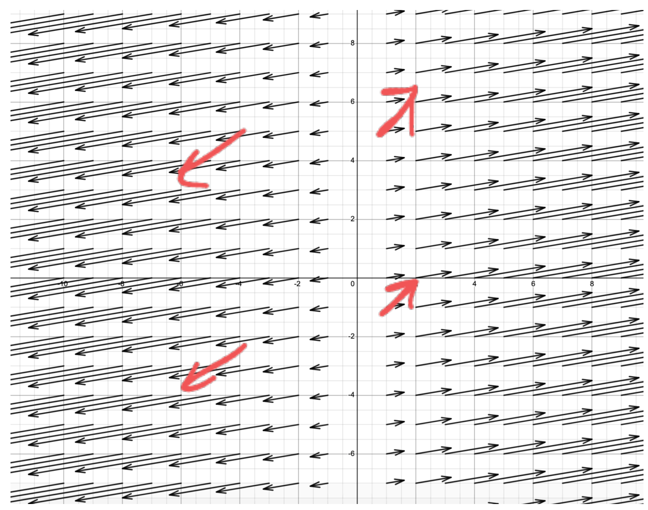
(4)



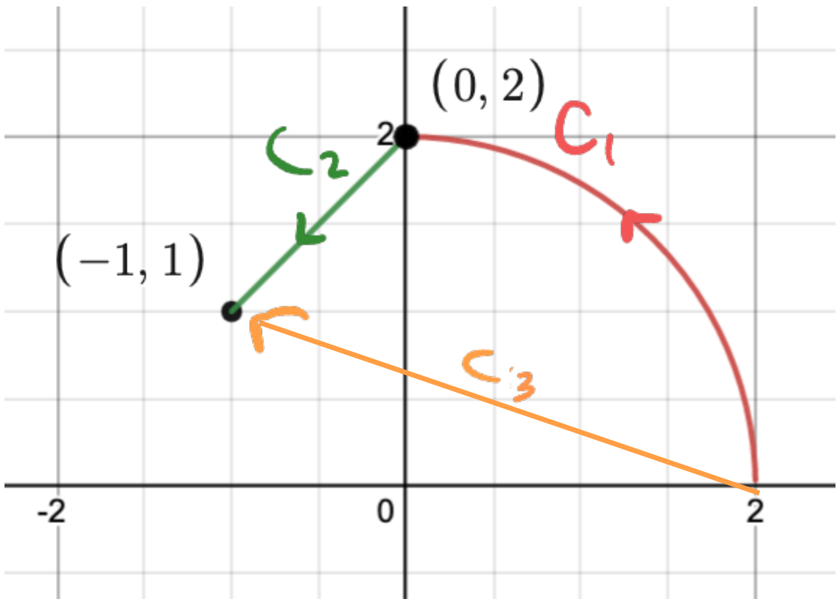
(5)



(6)



(2) Given $\vec{F}(x,y)=\langle x^2, y^2\rangle$ and the piecewise smooth path C given by the quarter circle of radius 2, traveled from (2,0) to (0,2), followed by the line segment from (0,2) to (-1,1) (30 points)



$\frac{\partial f}{\partial x} = x^2$

$f(x,y) = \frac{1}{3}x^3 + c(y)$

$\frac{\partial f}{\partial y} = c'(y) = y^2$

$c(y) = \frac{1}{3}y^3$

$f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$ check

a) Find the potential function $f(x,y)$ such that $\vec{\nabla} f(x,y) = \vec{F}(x,y)$

b) Find $\int_C \vec{F} \bullet d\vec{r}$ using two different methods. Name methods.

1) Using Fundemental thm $\int_C \vec{F}\cdot d\vec{r} = f(-1,1) - f(2,0) = \boxed{-\frac{8}{3}}$

2) since $\vec{F}$ conservative, we can use a simpler path

C_3: Line segment $(2,0) \to (-1,1)$ $x = 2-3t$, $y = t$, $0 \le t \le 1$

$\vec{F} = \langle (2-3t)^2, t^2 \rangle$

$\vec{r}\,' = \langle -3, 1 \rangle$

$\vec{F}\cdot\vec{r}\,' = -3(2-3t)^2 + t^2$

$\int_0^1 -3(2-3t)^2\,dt + \int_0^1 t^2\,dt$

$u = 2-3t$

$\frac{(2-3t)^3}{3}\Big]_0^1 + \frac{1}{3}$

$-\frac{1}{3} - \frac{8}{3} + \frac{1}{3} = \boxed{-\frac{8}{3}}$

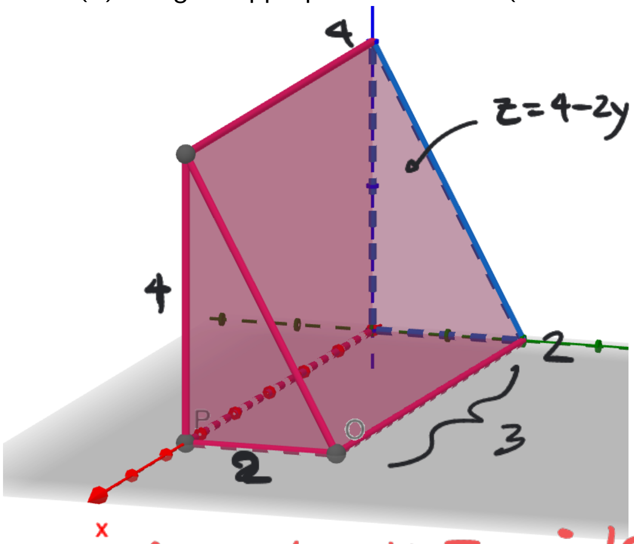
3) can also do directly

Or directly

$\int_{C_1} + \int_{C_2}$

(3) Evaluate the flux integral $\iint_S \vec{F}\bullet d\vec{S} = \iint_S \vec{F}\bullet\vec{n}\,dS$ where $\vec{F}(x,y,z)=\langle x,x,z\rangle$ and S is the closed surface formed by the plane $2y+z=4$ the coordinate planes. (outward unit normals), two ways: $x=3$

(a) directly, and (16 points)
(b) using an appropriate theorem. (name the theorem) (10 points)



Divergence Thm

$\text{div}\,\vec{F} = 2$

$\iiint_E 2\,dV = 2\text{ volume}$

$= 2\cdot\frac{1}{2}\cdot 4\cdot 2\cdot 3$

$\boxed{24}$

Need all 5 sides.

Directly

Front $x=3$

$\vec{F} = \langle 3,3,z\rangle$

$\vec{\nabla}G = \langle 1,0,0\rangle$

Flux $= \iint_D 3\,dA = 3(\text{Area})$

$= 3\cdot\frac{1}{2}\cdot 2\cdot 4$

$= 12$

Back $x=0$

$\vec{F} = \langle 0,0,z\rangle$

$-\vec{\nabla}G = \langle -1,0,0\rangle$

$\to 0$

Left $y=0$

$\vec{F} = \langle x,x,z\rangle$

$-\vec{\nabla}G = \langle 0,-1,0\rangle$

$\iint_D -x\,dA$

$\int_0^3\int_0^4 -x\,dz\,dx$

$\int_0^3 -4x\,dx$

$-2x^2\Big]_0^3$

-18

Bottom $z=0$

$\vec{F} = \langle x,x,0\rangle$

$-\vec{\nabla}G = \langle 0,0,-1\rangle$

$\to 0$

Top $z=4-2y$

$\underbrace{z+2y-4}_{G}=0$

$\vec{F} = \langle x,x,4-2y\rangle$

$\vec{\nabla}G = \langle 0,2,1\rangle$

$\vec{F}\cdot\vec{\nabla}G = 2x+4-2y$

$\int_0^3\int_0^2 (2x+4-2y)\,dy\,dx$

$\int_0^3 2xy+4y-y^2\Big]_0^2 dx$

$\int_0^3 4x+4\,dx$

$2x^2+4x\Big]_0^3$

$18+12 = 30$

$\text{Flux} = 12+0-18+0+30 = \boxed{24}$

Answers to (a) and (b) should match.

(4) Given $\vec{F}(x,y,z) = 2xy^2z\ \vec{i} + 2x^2yz\ \vec{j} + (x^2y^2 - 2z)\ \vec{k}$ and C is the curve given by $\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle, 0 \le t \le 2\pi$ find $\int_C \vec{F} \bullet d\vec{r}$ using any appropriate method. (name method) Show all steps of integration (12 points)

Directly $\vec{F} = \langle 2xy^2z,\ 2x^2yz,\ x^2z^2 - 2z \rangle$

$\vec{F} = \langle 2\cos t \sin^3 t,\ 2\cos^2 t \sin^2 t,\ \cos^2 t \sin^2 t - 2\sin t \rangle$

$\vec{r}' = \langle -\sin t,\ \cos t,\ \cos t \rangle$

$\vec{F} \cdot \vec{r}' = -2\cos t \sin^4 t + 2\cos^3 t \sin^2 t + \cos^3 t \sin^2 t - 2\cos t \sin t \rangle$

$= -2\cos t \sin^4 t + 3\cos^3 t \sin^2 t - 2\cos t \sin t$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} -2\cos t \sin^4 t\, dt + \int_0^{2\pi} \cos^3 t \sin^2 t\, dt - 2\int_0^{2\pi} \cos t \sin t\, dt$

$u = \sin t$ $\qquad$ $u = \sin t$

$-2\ \frac{\sin^5 t}{5}\Big]_0^{2\pi}$ $\qquad + \int_0^{2\pi} \cos t(1 - \sin^2 t)\sin^2 t\, dt$

$u = \sin t$

$0 \quad + \int_0^0 u^2 - u^4 \quad - \frac{\sin^2 t}{2}\Big]_0^{2\pi}$

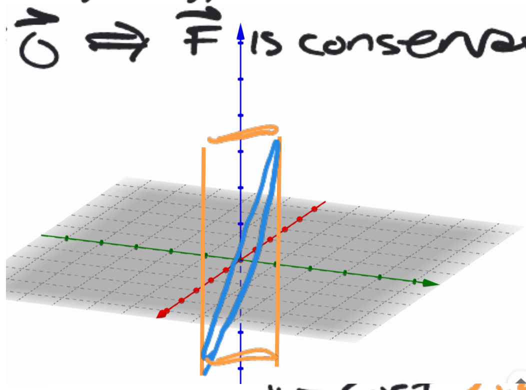
$0 \quad + \quad 0 \quad\quad 0 \quad = 0$

Stokes

curl $\vec{F}$:
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2z & 2x^2yz & x^2y^2 - 2z \end{vmatrix}$$

$\langle 2x^2y - 2x^2y,\ -(2xy^2 - 2xy^2),\ 4xyz - 4xyz \rangle$

$\vec{0} \Rightarrow \vec{F}$ is conservative



$x = \cos t$, $y = \sin t$ $\Big\}\ x^2 + y^2 = 1$

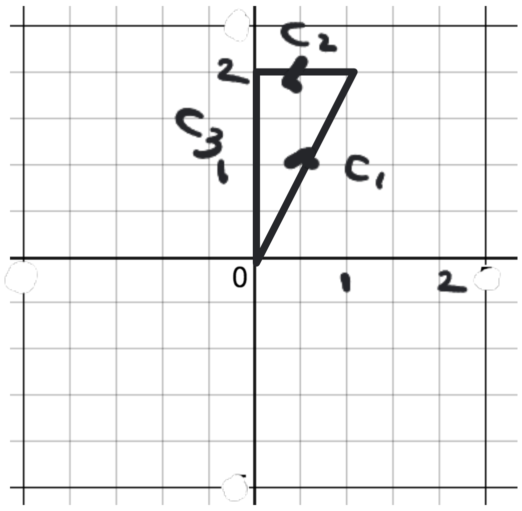
$y = \sin t$, $z = \sin t$ $\Big\}\ y = z$

$\vec{F}$ conservative, line integral around closed path is $\boxed{0}$.

(5) Find the work done by the vector field $\vec{F}(x,y) = <\sin x \cos y,\, xy + \cos x \sin y>$ in moving an object around a triangular path along $y = 2x$ from (0,0) to (1,2), (1,2) to (0,2) and returning to (0,0).

(12 points)

(use any appropriate method)



Green's Theorem

$$\frac{\partial Q}{\partial x} = y - \sin x \sin y$$

$$\frac{\partial P}{\partial y} = -\sin x \sin y$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D y\, dA = \int_0^2 \int_0^{\frac{1}{2}y} y\, dx\, dy = \int_0^2 \frac{1}{2} y^2\, dy = \frac{1}{6} y^3 \Big]_0^2 = \frac{4}{3}$$

Directly   Long way

C_1: $(0,0) \to (1,2)$

$x = t$

$y = 2t$

$0 \le t \le 1$

$\vec{F} = \sin t \cos 2t,\ 2t^2 + \cos t \sin 2t$

$\vec{r}\,' = <1, 2>$

$\int_0^1 \sin t \cos 2t + 4t^2 + 2\cos t \sin 2t$

yuk!

$\int_0^1 \sin t (2\cos^2 t - 1) + 4t^2 + 4\cos^2 t \sin t)\, dt$

$\int_0^1 (6\cos^2 t \sin t - \sin t + 4t^2)\, dt$

$-\frac{6\cos^3 t}{3} + \cos t + \frac{4}{3} t^3 \Big]_0^1$

$-\frac{6\cos^3 1}{3} + \cos 1 + \frac{4}{3} - (-2 + 1)$

$-\frac{6\cos^3 1}{3} + \cos 1 + \frac{4}{3} + 1$

C_2: $(1,2) \to (0,2)$

$x = 1 - t$

$y = 2$

$\vec{r}\,' = <-1, 0>$

$\vec{F} = <\sin(1-t), \cos 2, \ldots>$

$-\int_0^1 \sin(1-t) \cos 2\, dt$

$-\cos(1-t)\cos 2 \Big]_0^1$

$-\cos 2 + \cos 1 \cos 2$

C_3: $<0,2> \to <0,0>$

$x = 0$

$y = 2 - 2t$

$\vec{r}\,' = <0, -2>$

$\vec{F} = <0, \sin(2-2t)>$

$\int_0^1 -2\sin(2-2t)\, dt$

$-\cos(2-2t) \Big]_0^1$

$-1 + \cos 2$

It can be shown that the sum of these 3 is 4/3

(6)

(12 points)

Find the work done by the vector field $\vec{F}(x,y,z) = < y, z, x >$ in moving an object along the curve $\vec{r}(t) = \left\langle \sqrt{t}, \frac{1}{\sqrt{t}}, t \right\rangle$; $1 \le t \le 9$ using any appropriate method.

$\vec{F}$ not conservative so direct is only approach

$\vec{r} = < t^{1/2}, t^{-1/2}, t >$

$\vec{r}\,' = < \frac{1}{2}t^{-1/2}, -\frac{1}{2}t^{-3/2}, 1 >$

$\vec{F} = < t^{-1/2}, t, t^{1/2} >$

$\vec{F} \cdot \vec{r}\,' = \frac{1}{2}t^{-1} - \frac{1}{2}t^{-1/2} + t^{1/2}$

$\int_1^9 \left(\frac{1}{2}\frac{1}{t} - \frac{1}{2}t^{-1/2} + t^{1/2}\right)dt$

$\frac{1}{2}\ln|t| - t^{1/2} + \frac{2}{3}t^{3/2} \Big]_1^9$

$\frac{1}{2}\ln 9 - 3 + 18 + 1 - \frac{2}{3}$

$\frac{1}{2}\ln 9 + 16 - \frac{2}{3}$

$\boxed{\frac{1}{2}\ln 9 + \frac{46}{3}}$